TWO-STEP MATRIX ANALYSIS OF PRESTRESSED CABLE NETS

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We consider the behaviour of the cable-net shown in Fig 1 when arbitrary vertical forces are applied to the nodes. According to linear-algebraic analysis this net has 1 state of self-stress and 12 modes of inextensional deformation. We show that the applied loading may be decomposed into two parts. The first part does not excite any of the inextensional modes, while the second part is carried by the net through geometry changes associated with these inextensional modes. The response of the net is linear to each of these separate parts of the loading, and the analysis is done by means of 20×20 square matrices of full rank. We compute displacements for three different loading cases. We discuss, briefly, interactive effects between the two types of behaviour, and non-linearities due to stretching of the cables during 'large' deflection of the 'inextensional' modes; and we show that both of these refinements can be accommodated by means of rapidly converging iterative calculations.

I INTRODUCTION

The cable-net shown in Fig 1 consists of two sets of cables which are slung between rigid abutments, connected at the nodes, and pre-tensioned against each other. Cable nets of this sort, composed of quadrangular cells, are considerably less rigid than networks composed entirely of triangular cells. Thus, if a vertical load is applied to any one node of the net, it is carried mainly through the effects of geometrical distortion of the net. This may be demonstrated easily by means of a simple physical model. The net is relatively 'soft' in response to loads of this sort. It is also found by experiment that the relationship between the applied load and the displacement which it produces is non-linear.

The usual scheme of calculation of a cable net (see, e.g., Refs 1–7) envisages the assembly as multi-degree-of-freedom non-linear system. Several numerical schemes have been devised for solving the non-linear equations; but they all require many iterations and are consequently expensive to run on the computer.

In this paper we are concerned with a different kind of scheme for computation of the static response of a cable net to applied loading. The starting-point of our analysis is that an investigation of the 'equilibrium' and 'compatibility' equations of a net in its initial, unloaded configuration by means of classical linear algebra reveals that the assembly is a *pre-stressable mechanism* with a large number of inextensional modes of deformation. (Thus the example in Fig 1 has a single state of self-stress and 12 independent inextensional modes.) This leads directly to the idea that there are two distinct types of behaviour within the assembly. First, some patterns of loading are carried by the assembly as an ordinary structural framework, without excitation of any of the inextensional modes: the loads are balanced by changes of tension in the various cable-segments. Second, other patterns of loading are carried by virtue of the geometry-changes associated with the inextensional modes, without any change of tension in the cable-segments.

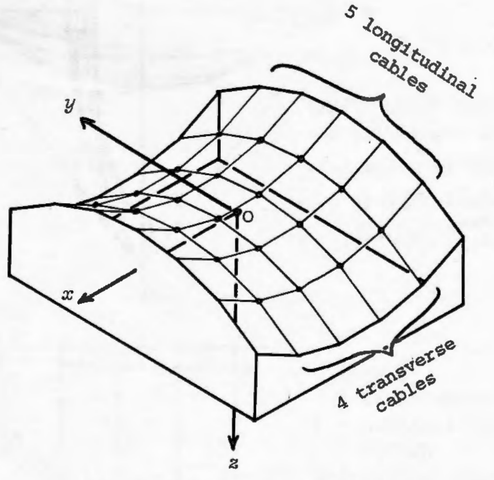


Fig 1 Saddle-shaped cable net

This scheme of classification was described first in Ref 8, where it was shown that the assembly is 'stiff' and 'soft' respectively in its two distinct types of action. It was also demonstrated that the two different types of action constitute two separate and distinct *linear* systems operating side-by-side (for sufficiently small loads, at least), each with its own eigenmodes.

For the past year we have been pursuing these ideas further, with a view to computing the behaviour of arbitrary cable nets under arbitrary static loading. We have developed computer programs which use linear algebra to generate, automatically, the states of self-stress and the inextensional modes of a given network; and we are developing other programs which perform the structural analysis of the two distinct kinds on the basis of this information.

In the present paper we give examples of the response of the net in Fig 1 to three different patterns of applied loading.

Our treatment is primarily on the lines of separate investigations of the two types of behaviour described above; but we also discuss the way in which the two types of behaviour can interact, and we introduce a simple way by which nonlinear effects due to relatively large deflection of the 'inextensional' modes may be computed. In these phases of the work it is sometimes necessary to adopt an iterative scheme of computation. However, the number of iterations required in practice appears to be very small.

It is interesting to note that the first treatment of the linear algebra of cable nets was given in a section of a paper by Buchholdt, Davies and Hussey (Ref 1). The section was self-contained, because the authors evidently saw no way of using to advantage the information generated by linear algebra in the general non-linear multi-degree-of-freedom scheme of computation which they described in the subsequent part of the paper. We have accepted the challenge implicit in the work of Ref 1; and we are now able to show how the results of the linear-algebraic approach to cable nets can be used in what promises to be an extremely efficient scheme for the computation of the behaviour of the nets.

Both Møllmann (Ref 5) and Irvine (Ref 9) have considered the behaviour of cable nets of the type shown in Fig 1, treating the assembly as a continuum governed by differential equations. The idea of decomposing the applied loading into two distinct parts does not appear in such a scheme.

II INEXTENSIONAL MECHANISMS AND 'FITTED' LOADS

For the sake of definiteness we consider a cable net consisting of 5 longitudinal 'sagging' cables and 4 transverse cables, as shown in Fig 1. The nodes all lie on the surface

$$\frac{z}{h} = -\frac{1}{6}\left(\frac{x}{\ell}\right)^2 + \frac{1}{9}\left(\frac{y}{\ell}\right)^2 \qquad \ldots\ldots \quad 1,$$

in the cartesian space whose axes are shown. The longitudinal and transverse cables lie in planes $y = 0, \pm\ell, \pm 2\ell$ and $x = \pm\frac{1}{2}\ell, \pm\frac{3}{2}\ell$ respectively, and the rigid abutments are in planes $y = \pm 3\ell$ and $x = \pm\frac{5}{2}\ell$. The 'dip' of the longitudinal cables from the abutment-points to the lowest nodes is equal to h, and so is the 'rise' of the transverse cables.

It is easy to verify that the net has a single state of self-stress (i.e. a state of stress in equilibrium with zero external load) in which the horizontal component of tension in every segment of the longitudinal cables is T_0, and in every segment of the transverse cables is $1.5T_0$. This produces a vertical reaction of $T_0 h/3\ell$ between the cables at each node of the net. The magnitude of T_0 may be altered by a tensioning device at the abutments. There is only *one* state of self-stress in the sense that, for the given configuration, the tension in *every* bar may be determined by the equations of equilibrium at the joints as soon as the tension in any *one* bar is given. (The net is similar to the one described in Ref 8, except that the geometry has been altered in detail to give different levels of self-stress in the two sets of cables.)

The first step in the linear-algebraic analysis of the net is to apply the general version of 'Maxwell's rule' (Ref 10):

$$3j - b = m - s \qquad \ldots\ldots \quad 2,$$

where j is the number of nodes or 'joints' (excluding abutment points),
b is the number of cable-segments (or 'bars': we assume throughout that every cable-segment is in tension),
m is the number of modes of inextensional distortion of the assembly, and
s is the number of states of self-stress.

Here $j = 20$, $b = 49$ and $s = 1$; and so

$$m = 12 \qquad \ldots\ldots \quad 3.$$

The form of the inextensional modes for this type of net (see Ref 8) is very simple. A typical mode (Fig 2) involves equal alternate up- and down-displacement at the four corner nodes of an interior cell, while all other nodes are fixed. In Fig 2 the net has been drawn as if it were plane, for the sake of clarity, but the pattern of *vertical* displacements is the same for the plane and the original net (cf. Ref 8). The four nodes of the original net also have components of displacement in the x- and y-directions; but these need not concern us here, as will be explained below. These modes on the pattern of Fig 2 are inextensional in the sense that each cable-segment suffers zero first-order extension when the displacement takes place. For this net the

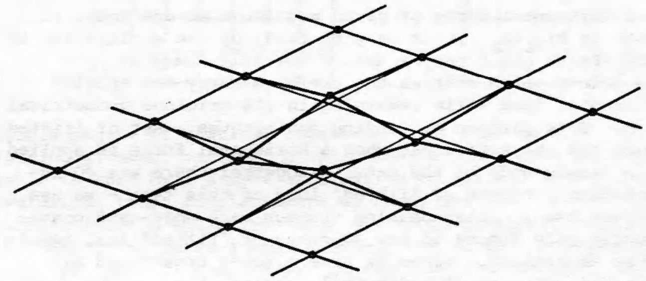


Fig 2 Inextensional mode of deformation

second-order changes in length are in fact non-zero, and so the modes are strictly inextensional only for sufficiently *small* displacements. We shall discuss the mechanical implications of this geometric feature in Section V. (Maxwell's rule, being rooted in the linear algebra of infinitesimal displacements, does not distinguish between 'free' and 'incipient' modes of inextensional displacement: see, e.g., Refs 11,12.)

The given net has exactly 12 'interior' cells. The pattern of Fig 2 may be applied at each cell in turn, making a total of 12 inextensional modes of deformation. The 12 modes are linearly independent in terms of their components of displacement, and thus any inextensional deformation of the assembly may be described as a linear combination of these 12 modes. (It is easy to show from Eqn 2 that in a more general net of the same type having m×n cables there are (m-1)×(n-1) inextensional modes on the pattern of Fig 2.)

Now the most general loading which can be applied to the net of Fig 1 has 3 components of force at each of the 20 nodes: the 'load space' has a dimension of 60. The corresponding 'displacement space' also has a dimension of 60, since there are 3 components of displacement at each joint. The linear-algebraic analysis of the net (which will be described in full elsewhere) indicates that the 60-dimensional displacement space consists of the 12-dimensional sub-space of *inextensional* displacements (described above), together with a 48 (=60 − 12) -dimensional space of *extensional* displacements. An extensional displacement of the assembly is like the deformation of an ordinary triangulated framework, in which the displacements of the joints are a direct kinematic consequence of the extensions of the bars. The main difference between a cable net and an 'ordinary' framework is that if we are to discuss the extensional modes of a cable net, we must make sure that the applied pattern of loading does not 'excite' any of the inextensional modes of the assembly. (It is necessary to check, of course, that all of the cable-segments remain in tension under a given loading.) In the language of linear algebra, the patterns of loading associated with the purely extensional modes must be *orthogonal* to each of the 12 inextensional modes. These 'fitted' loads (as Vilnay, Ref 13, calls them) fill the 48-dimensional sub-space of loads which is orthogonal to the 12-dimensional sub-space of inextensional displacements.

For the net of Fig 1 it is not difficult to obtain, by inspection, the 48 independent sets of 'fitted load' which are orthogonal to, and so do not excite, the inextensional modes. Consider first an isolated cable which is subjected

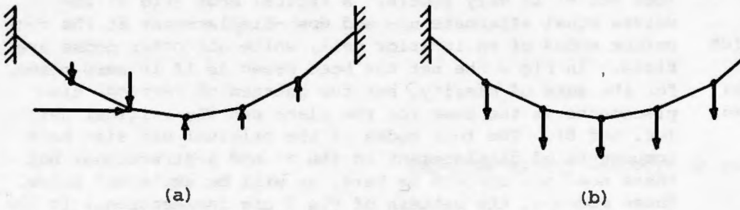


Fig 3 'Fitted' loads on single cable

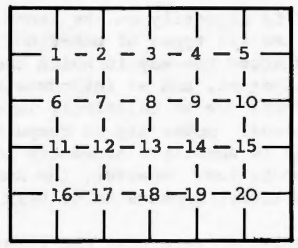


Fig 4 Numbering scheme for nodes, cf Fig 1

to a horizontal force of given magnitude at one node, as shown in Fig 3a. It is easy to find, by the application of equilibrium equations, a set of vertical loads at the joints which enables the cable to carry the applied horizontal load while remaining in its original geometrical form. This pattern of loading constitutes a set of 'fitted' loads for the entire net when a horizontal force is applied at a single node of the net. Altogether there are 40 independent patterns of 'fitted' load of this type - we use in turn the 2 cables passing through each node - and consequently only 8 more of the 48 cases of 'fitted' load remain to be determined. Since we have already considered all possible patterns of horizontal loading, these 8 cases can involve only the application of vertical forces to the nodes.

Figure 3b shows an isolated cable of the net under a set of *equal* vertical loads at the joints: when the magnitude of one of these has been assigned, the remainder follow from the equilibrium equations of the joints. In this way it seems that there will be 9 'fitted' loads of this pattern - since the net consists of 5 + 4 = 9 cables - rather than the 8 which we are seeking. This paradox is resolved by the observation that if all 5 longitudinal cables were equally loaded as in Fig 3b, there would then be equal loads on all of the nodes of the net; which would in fact be indistinguishable from a case in which all 4 transverse cables were equally loaded. It follows from this that the complete range of vertical load cases which are orthogonal to all 12 inextensional mechanisms is spanned by 'fitted' loads as in Fig 3b on any 8 of the 9 cables: it is immaterial which of the 9 is omitted for this purpose.

III 'FITTED' LOADS AND 'PRODUCT FORCES'

So far we have discussed the space of external loads as being 60-dimensional. For the sake of compactness let us now confine our attention to the case in which all of the loads applied at the joints are *vertical*. In this way we shall be concerned only with a 20-dimensional sub-space of the load space. Let us adopt the nodal numbering system shown in Fig 4. The eight 'fitted' loading cases described above may be represented by the first 8 columns of the 20×20 matrix in Eqn 4. (Cable 3-18 has been taken as the odd one.) The magnitude of any 'fitted' load is, of course, indeterminate, so it is convenient to use 1's in the 8 columns; the magnitudes are the coefficients $\alpha_1 \ldots \alpha_8$ in the postmultiplying column vector. Equation 4 states that any arbitrary loading state - here represented by a 20-element column vector on the right - can be described as the sum of the 20 column vectors of the square matrix after each has been multiplied by a coefficient α or β. If the equations can be solved uniquely, then $\alpha_1 \ldots \alpha_8$, $\beta_1 \ldots \beta_{12}$ are the components of 20 'standard' loading cases which are found in the given loading.

We have not yet discussed the 12 remaining columns in the 20×20 matrix of Eqn 4. What loading conditions do they represent? Following the preceding discussion, they must be 12 loading cases which *do* excite the inextensional modes, in contrast to the 8 'fitted' loading cases which do not.

Figure 5 shows part of a plane, square grid of cables under tension T_0 and $1.5T_0$, respectively, onto which has been imposed a typical unit case of inextensional displacement. It is clear that if $T_0 \neq 0$ some external forces are needed to maintain equilibrium. In Fig 5 the grid has been drawn plane since this simplified arrangement correctly gives (as may be shown easily; cf. Ref 8) the *vertical* forces which are required in the actual net when the inextensional mode occurs. This scheme does not include, of course, the associated horizontal forces which are required for equilibrium in the actual net; but this does not matter in the present problem, where we are considering only vertical forces, since any horizontal forces can be countered by superposition of the 40 discarded 'fitted' load cases. (In the case of a *shallow* net ($h \ll 5\ell$) the horizontal forces required for equilibrium would be small anyhow.)

The vertical forces shown in Fig 5 have been found by summing the forces needed for equilibrium of the cables separately. The forces needed for equilibrium of a single cable are found by using the equilibrium equations of the nodes in the distorted configuration. This is a *linear* calculation, since the displacements are assumed to be small; and the pretension in the elastic cables does not change in an inextensional mode of deformation. The forces in Fig 5 have been given numerical values for the sake of simplicity, but it is easy to show that the unit of these forces is $T_0 w/\ell$, where $\pm w$ is the vertical displacement of the nodes.

$$
\left[\begin{array}{cccccccc:cccccccccccc}
1 & & & & 1 & & & & p & -q & & & -1 & & & & & & & \\
1 & & & & & 1 & & & -p & p & -q & & 1 & -1 & & & & & & \\
1 & & & & & & & & q & -p & p & -q & & 1 & -1 & & & & & \\
1 & & & & & & 1 & & & q & -p & p & & & 1 & -1 & & & & \\
1 & & & & & & & 1 & & & q & -p & & & & 1 & & & & \\
 & 1 & & & 1 & & & & -p & q & & & p & -q & & & -1 & & & \\
 & 1 & & & & 1 & & & p & -p & q & & -p & p & -q & & 1 & -1 & & \\
 & 1 & & & & & & & -q & p & -p & q & q & -p & p & -q & & 1 & -1 & \\
 & 1 & & & & & 1 & & & -q & p & -p & & q & -p & p & & & 1 & -1 \\
 & 1 & & & & & & 1 & & & -q & p & & & q & -p & & & & 1 \\
 & & 1 & & 1 & & & & 1 & & & & -p & q & & & p & -q & & \\
 & & 1 & & & 1 & & & -1 & 1 & & & p & -p & q & & -p & p & -q & \\
 & & 1 & & & & & & & -1 & 1 & & -q & p & -p & q & q & -p & p & -q \\
 & & 1 & & & & 1 & & & & -1 & 1 & & -q & p & -p & & q & -p & p \\
 & & 1 & & & & & 1 & & & & -1 & & & -q & p & & & q & -p \\
 & & & 1 & 1 & & & & & & & & 1 & & & & -p & q & & \\
 & & & 1 & & 1 & & & & & & & -1 & 1 & & & p & -p & q & \\
 & & & 1 & & & & & & & & & & -1 & 1 & & -q & p & -p & q \\
 & & & 1 & & & 1 & & & & & & & & -1 & 1 & & -q & p & -p \\
 & & & 1 & & & & 1 & & & & & & & & -1 & & & -q & p
\end{array}\right]
\cdot
\left[\begin{array}{c}
\alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_8 \\ \hdashline \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_{12}
\end{array}\right]
= F \cdot
\underset{\text{(i)}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}}\ ,\quad
\underset{\text{(ii)}}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ 1 \\ 1 \\ 1 \\ \\ \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}}\ ,\quad
\underset{\text{(iii)}}{\begin{bmatrix} \\ \\ \\ 1 \\ 1 \\ \\ \\ \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ 1 \\ 1 \\ \\ \\ \\ \end{bmatrix}}
\qquad \ldots\ldots\ 4.
$$

Key: $p=7.5$, $q=1.5$

There is a set of loads of this kind associated with each of the 12 distinct inextensional modes; and it is these 12 sets which appear in columns 9-20 of the matrix in Eqn 4. We call these the 'product forces' of the net. This is not an ideal descriptive term, but it does express the idea that the forces concerned are a sort of product of the

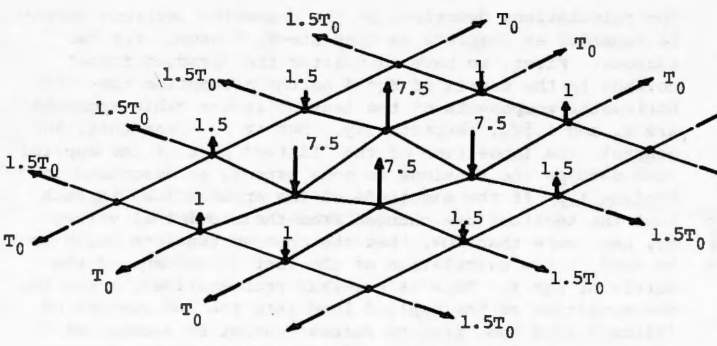
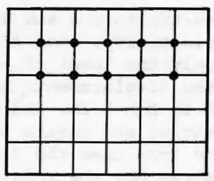
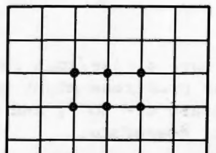
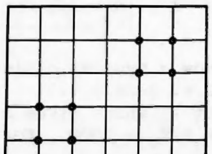


Fig 5 'Product' forces for mode of Fig 2

Fig 6 Three patterns of loading







state of prestress and an inextensional mode.

The matrix is in fact of full rank, and so the equations have a unique solution for any given loading case. Three particular loading cases (i-iii) are given in Eqn 4, corresponding to the three loading conditions shown in Fig 6. In each case a node which is loaded carries a vertical force of magnitude F in the downward direction ($z > 0$) of Fig 1.

The solution of Eqn 4 is given in Eqn 5. It is easy to see that case (i) is a simple combination of 2 'fitted' loads, so that in particular $\beta_1 \dots \beta_{12} = 0$. In case (ii) the applied load decomposes into 8 'fitted' loads and 8 'product forces', though the latter have small magnitude in comparison with the two leading 'fitted' loads. Lastly, in case (iii), it is the 'product forces' which have the dominant coefficients.

IV DISPLACEMENTS OF THE CABLE NET

Having decomposed a given set of applied loads into 'fitted' loads and 'product forces', we can now examine the displacement of the assembly under each of the 20 component loading conditions separately, and then take the sum. This operation is shown in Eqn 6. Here $w_1 \dots w_{20}$ are the vertical displacements of the 20 nodes. The 20×20 matrix has 8 columns corresponding to the 'fitted' loads and 12 for the inextensional mechanisms. The latter are self-explanatory in terms of Fig 2. The derivation of the first 8 columns is straightforward, and we do not give full details here. Briefly, we first calculate the tension in every cable segment when a given fitted load is applied to the cable net, either by using the method given in Ref 8 or by a virtual work calculation in which the state of prestress is used as a 'dummy load' condition: the problem involves a single degree of statical indeterminacy. Then we find the (strictly, mean) vertical displacement of the nodes of each cable in turn by a simple application of virtual work. (In evaluating these displacements we have, for the sake of computational simplicity, taken the cross-sectional area of the horizontal cable-segments as A_0, and of the inclined segments as $A_0\sec^3\theta$, where θ is the angle of inclination. In this way the calculation becomes independent of the value of h/ℓ. E is the Young's modulus of the material of the cables.)

The matrix in Eqn 6 is post-multiplied by a column vector which contains $\alpha_1 \dots \alpha_8$, $\beta_1 \dots \beta_{12}$ as before, but now with

$$\begin{bmatrix}\alpha_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \alpha_8\\ \hline \beta_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \beta_{12}\end{bmatrix} = \frac{F}{10}\cdot \begin{bmatrix}10.\\ 10.\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix};\quad \begin{bmatrix}1.92\\ 8.55\\ 8.55\\ 1.92\\ -5.49\\ -0.09\\ -0.09\\ -5.49\\ \hline 0.52\\ 0.23\\ -0.23\\ -0.52\\ 0\\ 0\\ 0\\ 0\\ -0.52\\ -0.23\\ 0.23\\ 0.52\end{bmatrix};\quad \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 5.\\ 5.\\ 5.\\ 5.\\ \hline -1.86\\ -3.72\\ -3.72\\ -1.86\\ -3.39\\ -6.78\\ -6.78\\ -3.39\\ -1.86\\ -3.72\\ -3.72\\ -1.86\end{bmatrix} \quad \dots\dots\ 5.$$

$$\begin{bmatrix}w_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ w_6\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ w_{11}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ w_{16}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix} = \left[\begin{array}{cccccccc|cccccccccccc}
\kappa & \lambda & \lambda & \lambda & \mu & \nu & \nu & \nu & 1 & & & & & & & & & & & \\
\kappa & \lambda & \lambda & \lambda & \nu & \mu & \nu & \nu & -1 & 1 & & & & & & & & & & \\
\kappa & \lambda & \lambda & \lambda & \nu & \nu & \nu & \nu & & -1 & 1 & & & & & & & & & \\
\kappa & \lambda & \lambda & \lambda & \nu & \nu & \mu & \nu & & & -1 & 1 & & & & & & & & \\
\kappa & \lambda & \lambda & \lambda & \nu & \nu & \nu & \mu & & & & -1 & & & & & & & & \\
\hline
\lambda & \kappa & \lambda & \lambda & \mu & \nu & \nu & \nu & -1 & & & & 1 & & & & & & & \\
\lambda & \kappa & \lambda & \lambda & \nu & \mu & \nu & \nu & 1 & -1 & & & -1 & 1 & & & & & & \\
\lambda & \kappa & \lambda & \lambda & \nu & \nu & \nu & \nu & & 1 & -1 & & & -1 & 1 & & & & & \\
\lambda & \kappa & \lambda & \lambda & \nu & \nu & \mu & \nu & & & 1 & -1 & & & -1 & 1 & & & & \\
\lambda & \kappa & \lambda & \lambda & \nu & \nu & \nu & \mu & & & & 1 & & & & -1 & & & & \\
\hline
\lambda & \lambda & \kappa & \lambda & \mu & \nu & \nu & \nu & & & & & -1 & & & & 1 & & & \\
\lambda & \lambda & \kappa & \lambda & \nu & \mu & \nu & \nu & & & & & 1 & -1 & & & -1 & 1 & & \\
\lambda & \lambda & \kappa & \lambda & \nu & \nu & \nu & \nu & & & & & & 1 & -1 & & & -1 & 1 & \\
\lambda & \lambda & \kappa & \lambda & \nu & \nu & \mu & \nu & & & & & & & 1 & -1 & & & -1 & 1 \\
\lambda & \lambda & \kappa & \lambda & \nu & \nu & \nu & \mu & & & & & & & & 1 & & & & -1 \\
\hline
\lambda & \lambda & \lambda & \kappa & \mu & \nu & \nu & \nu & & & & & & & & & -1 & & & \\
\lambda & \lambda & \lambda & \kappa & \nu & \mu & \nu & \nu & & & & & & & & & 1 & -1 & & \\
\lambda & \lambda & \lambda & \kappa & \nu & \nu & \nu & \nu & & & & & & & & & & 1 & -1 & \\
\lambda & \lambda & \lambda & \kappa & \nu & \nu & \mu & \nu & & & & & & & & & & & 1 & -1 \\
\lambda & \lambda & \lambda & \kappa & \nu & \nu & \nu & \mu & & & & & & & & & & & & 1
\end{array}\right] \cdot \begin{bmatrix}A\alpha_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ A\alpha_8\\ \hline B\beta_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ B\beta_{12}\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}16.0\\ 16.0\\ 16.0\\ 16.0\\ 16.0\\ \hline 16.0\\ 16.0\\ 16.0\\ 16.0\\ 16.0\\ \hline -8.3\\ -8.3\\ -8.3\\ -8.3\\ -8.3\\ \hline -8.3\\ -8.3\\ -8.3\\ -8.3\\ -8.3\end{bmatrix} AF + \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \frac{BF}{10}}_{(i)};\ \underbrace{\begin{bmatrix}-9.41\\ -3.33\\ -3.23\\ -3.33\\ -9.41\\ \hline 6.69\\ 12.77\\ 12.87\\ 12.77\\ 6.69\\ \hline 6.69\\ 12.77\\ 12.87\\ 12.77\\ 6.69\\ \hline -9.41\\ -3.33\\ -3.23\\ -3.33\\ -9.41\end{bmatrix} AF + \begin{bmatrix}0.52\\ -0.29\\ -0.47\\ -0.29\\ 0.52\\ \hline -0.52\\ 0.29\\ 0.47\\ 0.29\\ -0.52\\ \hline -0.52\\ 0.29\\ 0.47\\ 0.29\\ -0.52\\ \hline 0.52\\ -0.29\\ -0.47\\ -0.29\\ 0.52\end{bmatrix} \frac{BF}{10}}_{(ii)};\ \underbrace{\begin{bmatrix}4.20\\ 4.20\\ -1.42\\ 4.20\\ 4.20\\ \hline 4.20\\ 4.20\\ -1.42\\ 4.20\\ 4.20\\ \hline 4.20\\ 4.20\\ -1.42\\ 4.20\\ 4.20\\ \hline 4.20\\ 4.20\\ -1.42\\ 4.20\\ 4.20\end{bmatrix} AF + \begin{bmatrix}-1.86\\ -1.86\\ 0\\ 1.86\\ 1.86\\ \hline -1.53\\ -1.53\\ 0\\ 1.53\\ 1.53\\ \hline 1.53\\ 1.53\\ 0\\ -1.53\\ -1.53\\ \hline 1.86\\ 1.86\\ 0\\ -1.86\\ -1.86\end{bmatrix} \frac{BF}{10}}_{(iii)} \quad \dots\ 6.$$

Key: $\kappa=20.15$, $\lambda=-4.15$, $\mu=10.54$, $\nu=-0.71$, $A=\ell^3/h^2A_0E$, $B=\ell/T_0$

multipliers A and B. The factor A involves the modulus of elasticity, etc., of the cable segments, while B involves only the level of prestress in the net. On the RHS of Eqn 6 the displacements have been evaluated from the values of α, β in Eqn 5 for the three loading cases, and with the extensional and inextensional moieties separately. Notice that in each case the displacement is proportional to F; but observe how the relative magnitude of the two parts depends in particular on the magnitude of the factor

$$\frac{B}{A} = \left(\frac{h}{\ell}\right)^2 \frac{A_0E}{T_0} \qquad \ldots\ldots \quad 7.$$

Now a typical cable-net might have a rise/span ratio of 0.1, i.e. here $h/5.5\ell = 0.1$; and the prestress might have a value of T_0 which gives a tensile strain $\varepsilon \simeq 10^{-3}$, and thus $A_0E/T_0 \simeq 1000$. For such a net, therefore,

$$\frac{B}{A} = 300 \qquad \ldots\ldots \quad 8.$$

Inspection of the numbers in the RHS columns of Eqn 8 with this value of B/A reveals that in loading case (ii)(Fig 6) the displacements due to extensional and inextensional effects are of the same order, while in case (iii) the inextensional effects are dominant.

Isometric plots of vertical displacement for loading cases (ii) and (iii) are shown in Fig 7. The nodes which carry forces are marked. The value of F in case (ii) has been set at 4/3 of that in case (iii), so that the *total* vertical load is the same in both cases: total load = 8F.

It is clear that the net deflects more when the load is disposed on a diagonal, as in (iii), than when it is applied centrally, as in (ii). The displacements corresponding to case (i) are not illustrated. They are much smaller in magnitude (see Eqn 6), since the inextensional modes are not excited.

V DISCUSSION

Our example provides a clear illustration of the way in which the two kinds of behaviour in cable nets may be analysed by means of linear algebra. For the sake of simplicity, and at the expense of some precision, we have dealt only with the 20-dimensional subspace of vertical loads and displacements. But the same principles apply equally to the full 60-dimensional version, and the overall results do not differ by much. As we mentioned earlier, our example has the convenient feature that we can find the columns of the relevant matrices by inspection, and at the expense of little effort. In a less straightforward practical problem, of course, the computer would *generate* the relevant matrices automatically from the data on the form of the net, by use of the program mentioned in Section I.

The calculations described in the preceding sections cannot be regarded as complete as they stand, however, for two reasons. First, we have calculated the 'product force' columns in the matrix of Eqn 4 on the assumption that the horizontal components of the tension in the cable segments are T_0 and $1.5T_0$, respectively. But it is clear that, in general, the imposition of the 'fitted' part of the applied load *changes* the tensions to some extent, as described in Section IV. If the magnitude of the applied load is such that the tensions are changed from their original values by, say, more than 20%, then the changed tensions ought to be used in the calculation of the last 12 columns of the matrix of Eqn 4. This is somewhat problematical, since the decomposition of the applied load into the two classes of 'fitted' load and 'product forces' cannot be accomplished until the matrix is known; and yet the columns cannot be finalised until the decomposition is known. Some simple iterative calculations which we have performed on these lines suggest that the process converges rapidly. For example, only one iteration is required to complete the calculation for our loading case (iii).

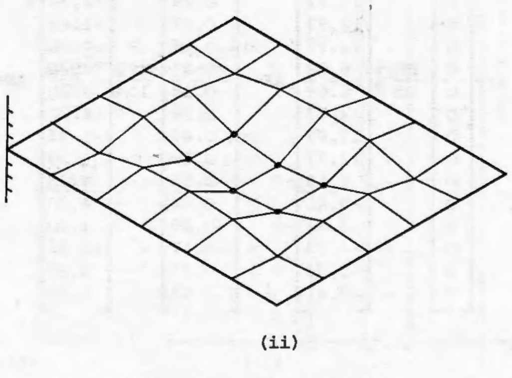
The second question concerning the correctness of the results comes from the observation that the 'inextensional' displacements are only truly inextensional for sufficiently small displacements.

Consider, for example, the cable shown in Fig 8a. The displacement shown has $w/\ell = 0.1$, so the two segments rotate by arcsin 0.1 = 5.74°. This involves an overall stretching of the cable which is equal to (sec 5.74° − 1) = 0.005 of the length of the two segments concerned; or 2/5 of this, i.e. 0.002, of the length of the entire cable. Thus, if the original level of prestress in the cable represents a strain of 0.001, the displacement shown in Fig 8a would treble the prestressing tension. In general, the load-displacement relationship is non-linear, since T increases with w.

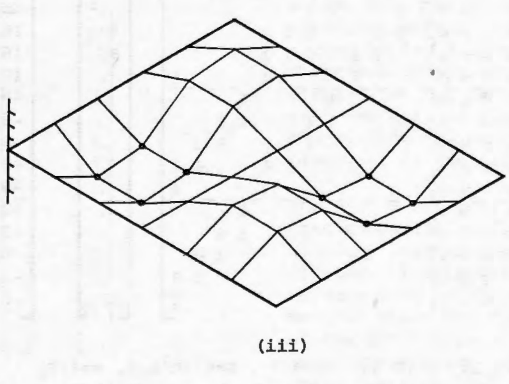
Now in relation to the transverse forces on the cable which are necessary to produce the displacement shown, the previous calculation of 'product forces' would be correct if only we could determine the *current* level of prestress, taking into account stretching effects of the kind just described. The calculation above suggests that we would need to investigate the angular rotation of all of the cable-elements in the net. It turns out, however, that this is not necessary; and that the calculation can be performed easily, in one step, as soon as the linearised computation has been performed.

Consider again the example shown in Fig 8a, with the cable at a prestressing tension T_0. Further, suppose that the original elastic cable is replaced by a *perfectly plastic* one having a *constant* yield tension T_0. When the transverse force P is now applied the tension remains at T_0 since T_0 is now a property of the cable. In this case,

$$P = 2T_0w/\ell \qquad \ldots\ldots \quad 9,$$



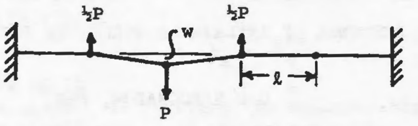
(ii)



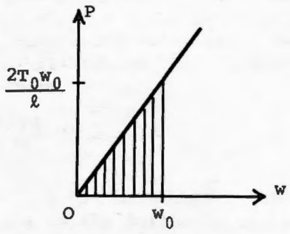
(iii)

Fig 7 Plot of vertical displacements, cf. Fig 6. Unit for vertical scale: $F\ell/10T_0$.

Fig 8 Computation of the extension of a cable



(a)



(b)

when we neglect terms of order $T_0(w/\ell)^3$. This linear relationship between P and w is shown in Fig 8b.

Now suppose that P has increased steadily until $w = w_0$. The total work done on the wire by the applied transverse load is equal to the shaded area, *viz.*

$$T_0 w_0^2/\ell \qquad \ldots\ldots \quad 10.$$

The only way in which the cable can absorb this work is by stretching; and since the tension remains constant at T_0, the overall extension e of cable is given by

$$e = (T_0 w_0^2/\ell) \div T_0 = \frac{w_0^2}{\ell} \qquad \ldots\ldots \quad 11.$$

The original length of cable is 5ℓ, and hence the overall strain ε in the cable due to the distortion from its straight original configuration is given by,

$$\varepsilon = 0.2(w_0/\ell)^2$$
$$= 0.002 \qquad \text{when } w_0/\ell = 0.1 \qquad \ldots\ldots \quad 12.$$

The result is the same as before.

We have used here the artifice of a 'perfectly plastic' cable in order to perform a *geometrical* calculation; and the result is therefore valid even when the cable is elastic.

The same kind of calculation can be used to find the increase in the level of pretension in an entire cable net which is subjected to a general pattern of loading. Thus, starting with the total work done by the 'product forces' according to the linear calculation (as described, e.g., in Eqns 4,6) and using the virtual-work method outlined in Section IV, we can derive a single factor by which the level of prestress would actually be increased if the calculated displacement were to occur. An important point here is that our cable net has only *one* state of self-stress, and so the non-linear behaviour is described very simply by means of an increase in this single quantity.

This calculation indicates that larger loads are needed to produce the deflections which were computed by the linearised scheme for given loads. It is therefore necessary to do an iteration in order to find the correct displacements for a given load. This iteration, however, is concerned only with the solution of a cubic equation in a single variable, and is very straightforward.

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